Modified Rubey's Law Accurately Predicts Sediment Settling Velocities

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Abstract. Modification of viscous drag and the pressure drag terms in Rubey's theoretic equation for settling velocity of sand grains with two empirically determined constants results in the following formula:

\[ V = \frac{\sqrt{6} Z \mu^2 + \frac{1}{4} WR^2 (\rho_{\text{particle}} - \rho_{\text{fluid}}) g - 3Z\mu}{WR \rho_{\text{fluid}}} \]

which predicts accurately the settling velocity of groups of sand grains measured with the Woods Hole rapid sediment analyzer. The following equation is a solution of Rubey's modified general equation that can be used to calculate the diameter of a particle or group of particles if the settling velocity for the particle is known:

\[ R = \frac{\sqrt{W^2 V^4 \rho_{\text{fluid}}^3 + 32Z (\rho_{\text{particle}} - \rho_{\text{fluid}}) \mu V g + WV^2 \rho_{\text{fluid}}}}{3(\rho_{\text{particle}} - \rho_{\text{fluid}}) g} \]

NOTATIONS

- \( V \), fall velocity, cm/sec;
- \( \rho \), mass density, g/cm³;
- \( R \), particle radius, cm;
- \( \mu \), dynamic viscosity, poise, dyne-sec/cm²;
- \( g \), gravitational acceleration, 980 cm/sec²;
- \( W \), pressure drag coefficient, dimensionless = .5305;
- \( Z \), viscous drag coefficient, dimensionless = .622;
- \( \phi \), -Logs, grain diameter, num.

INTRODUCTION

In 1933 W. W. Rubey analytically derived an equation to relate the settling velocity of sand grains to fluid and particle properties. His equation is analytically complete since it includes all of the forces acting on a falling particle. Rubey's theoretic equation generates a curve of the same shape as empirically determined settling velocities; however, it predicts settling times that are longer than the actual measured settling times for sand particles falling through 100 cm of water (Figure 1). This suggests that modification of Rubey's wholly theoretic equation by empirically determined viscous and pressure drag coefficients will result in a useful equation to determine settling velocities of sediment particles in any fluid. The purpose of this paper is to compute these drag coefficients.

A modification of Rubey's law for the fall velocity of spheres with two empirically determined constants accurately predicts the settling velocities of sand measured by the rapid sediment analyzer at Woods Hole. The modified equation will enable an investigator to use the rapid sediment analyzer for the determination of settling velocities for sand in water of any temperature, in fluids other than water, and for sediment with a specific gravity other than that of quartz. In addition this modified equation may prove useful for the calculation of the settling velocity of groups of sand grains in natural systems such as rivers.

MODIFIED RUBEY'S LAW

In its original form, Rubey's law predicts fall times that are longer than the fall times (for fall through 100 cm of water) measured with the sediment analyzer (Figure 1). Rubey's [1933] original formula is shown below.
Fig. 1. Grain diameter versus time to fall 100 cm in water at a temperature of 21-23°C.

\[ V = \frac{\sqrt{9a^2 + 4R^2(\rho_{\text{particle}} - \rho_{\text{fluid}})g}}{R \rho_{\text{fluid}}} - 3\mu \]  

Note that the curve for Rubey's law has exactly the same shape as that for the empiric data measured at Woods Hole [Schlee, 1956], but Rubey's law predicts fall times that are too long. The curve which nearly fits the data points is the curve generated by Rubey's law modified by empiric constants \( Z \) and \( W \). This modified form of Rubey's law can be stated as follows:

\[ V = \frac{\sqrt{9Z^2\mu^2 + \frac{4}{3}WR(\rho_{\text{particle}} - \rho_{\text{fluid}})g}}{WR \rho_{\text{fluid}}} - 3Z\mu \]  

where \( Z = .622 \) and \( W = .5305 \). A plot of fall velocities measured at Woods Hole versus fall velocities calculated with the modified form of Rubey's law shows an excellent correlation (Figure 2 and Table 1). The fall velocity for groups of sediment grains can now be computed for any fluid and for any temperature, provided that the particle density and radius and the fluid density and viscosity are known.

The following equation, which is a solution of the modified Rubey's law for particle radius, can be used with fall velocity data obtained from the Woods Hole sediment analyzer to determine particle size:

\[ R = \frac{\sqrt{W^2V^2\rho_{\text{fluid}} + 32Z(\rho_{\text{particle}} - \rho_{\text{fluid}})Vg}}{\frac{4}{3}(\rho_{\text{particle}} - \rho_{\text{fluid}})g} \]  

**DERIVATION OF MODIFIED RUBEY'S LAW**

The two equations above can be derived by writing an equation to balance the forces acting on a spheric particle falling through a fluid at terminal velocity. At terminal velocity the forces are in balance since the particle is not accelerating. Therefore the force balance equation can be written as follows:

- particle weight in fluid = viscous drag force + pressure drag force
particle weight in fluid = ½πR³(ρ_{partic} - ρ_{fluid})g
viscous drag force (Stokes' law) = 6πRμV
pressure drag force = -πR²V^2ρ_{fluid}

The pressure drag force is the product of the
projected area of the particle πR² and the
pressure acting on it due to its motion relative to
the fluid ρ_{fluid} V²/2 [Streeter, 1960]. The similarity
of shape between the curves for unmodified
Rubey's law and the data (Figure 1) led me to
believe that modification of the viscous drag and
the pressure drag by empirically determined
constants might lead to an accurate solution for
fall velocity. These constants probably account
for the lack of sphericity of the grains and non-
ideal behavior of the fluid. With this considera-
tion the force balance equation is written as
follows:

Particle weight in fluid = (Z) (viscous drag
force) + (W) (pressure drag force)

or

\[ \frac{1}{2} \pi R^3 (\rho_{partic} - \rho_{fluid})g = Z(6\pi R \mu V) + W(\pi R^2 V^2 \rho_{fluid}) \] (4)

The factor of ½ in the pressure drag term is
absorbed into the coefficient W to retain paral-
lelism with the equation derived by Rubey
[1933].

Equation 4 can now be simplified and solved
for either particle radius or velocity by the
quadratic formula yielding equations 5 and 2,
respectively.

**EVALUATION OF THE COEFFICIENTS Z AND W**

As indicated in Figure 1 the fall velocity of
particles of a diameter of -2φ to 0φ is con-
trolled almost entirely by pressure drag. Parti-
cles of diameter 3φ to 4φ are controlled almost
entirely by viscous drag (Stokes' law). In the
intermediate range both drag forces are im-
portant. The CDC 6600 computer at the Un-
iversity of Texas at Austin was used to determine
Z and W by trial and error in the following
manner. Equation 4 above was solved separately
for Z and for W in terms of all of the other
variables. A value was assumed for W, then Z's
were calculated for known diameters and their

-2φ to 0φ range. This process of successive
approximations was continued until both Z and
W converged to constants. These values were
then substituted into equation 2 and used to
compute the modified Rubey's Law curve.

<table>
<thead>
<tr>
<th>Woods Hole</th>
<th>Rubey's Law</th>
<th>Grain Diameter</th>
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</thead>
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<td>Velocity (cm/sec)</td>
<td>Diameter φ</td>
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<tr>
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Fig. 2. Fall velocities of particles measured at
Woods Hole by the Rapid Sediment Analyzer
versus fall velocities calculated with the modified
form of Rubey's Law. The straight line is the
line (at 45°) which would represent a perfect
correlation between the two sets of data.
(Figure 1). The close fit of the raw data from Woods Hole and the modified curve indicate that the values for constants Z and W are satisfactory.

CONCLUSIONS

Rubey's law modified with two empirical constants very accurately predicts fall velocities for groups of sand grains measured with the Woods Hole rapid sediment analyzer. This modified formula can now be used for calculation of particle size with fall time data from the sediment analyzer. The advantage of this method rather than the use of empirical data for particle size determination is that the determination can be made regardless of the water temperature in the settling tube, the viscosity of the fluid in the tube, or the relative densities of the fluid and the particles, as long as these values are known. This modified formula may also prove useful in predicting the fall velocity of groups of sediment grains in transport in such natural systems as river and wave transport.

Acknowledgments. I would like to thank Dr. Robert L. Folk and Dr. Earl F. McBride of the Department of Geological Sciences and Dr. Carl Morgan of the Department of Civil Engineering of the University of Texas at Austin for reading this manuscript critically and for their suggestions. The computation center of the University of Texas supplied access to the CDC 6600 computer and computer time, without which the study could not have been completed. Drafting was done by Judith Evelyn Watson.

REFERENCES


(Manuscript received March 19, 1969; revised April 23, 1969.)