Modified Rubey's Law Accurately Predicts Sediment Settling Velocities

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Abstract. Modification of viscous drag and the pressure drag terms in Rubey's theoretic equation for settling velocity of sand grains with two empirically determined constants results in the following formula

$$V = \frac{\sqrt{9Z^2\mu^2 + \frac{4}{3}WR^3(\rho_{\text{particle}} - \rho_{\text{fluid}})g} - 3Z\mu}{WR\rho_{\text{fluid}}}$$

which predicts accurately the setting velocity of groups of sand grains measured with the Woods Hole rapid sediment analyzer. The following equation is a solution of Rubey's modified general equation that can be used to calculate the diameter of a particle or group of particles if the settling velocity for the particle is known:

$$R = \frac{\sqrt{W^2V^4\rho^2_{\text{fluid}} + 32Z(\rho_{\text{particle}} - \rho_{\text{fluid}})\mu Vg} + WV^2\rho_{\text{fluid}}}{\frac{8}{3}(\rho_{\text{particle}} - \rho_{\text{fluid}})g}$$

NOTATIONS

V, fall velocity, cm/sec;
ρ, mass density, g/cm^2;
R, particle radius, cm;
μ, dynamic viscosity, poise, dyne-sec/cm^2;
g, gravitational acceleration, 980 cm/sec^2;
W, pressure drag coefficient, dimensionless = .5305;
Z, viscous drag coefficient, dimensionless = .622;
ϕ, $-\text{Log}_2$, grain diameter, mm.

INTRODUCTION

In 1933 W. W. Rubey analytically derived an equation to relate the settling velocity of sand grains to fluid and particle properties. His equation is analytically complete since it includes all of the forces acting on a falling particle. Rubey's theoretic equation generates a curve of the same shape as empirically determined settling velocities; however, it predicts settling times that are longer than the actual measured settling times for sand particles falling through 100 cm of water (Figure 1). This suggests that modification of Rubey's wholly theoretic equation by empirically determined viscous and pressure drag coefficients will result in a useful equation to determine settling velocities of sediment particles in any fluid. The purpose of this paper is to compute these drag coefficients.

A modification of Rubey's law for the fall velocity of spheres with two empirically determined constants accurately predicts the settling velocities of sand measured by the rapid sediment analyzer at Woods Hole. The modified equation will enable an investigator to use the rapid sediment analyzer for the determination of settling velocities for sand in water of any temperature, in fluids other than water, and for sediment with a specific gravity other than that of quartz. In addition this modified equation may prove useful for the calculation of the settling velocity of groups of sand grains in natural systems such as rivers.

MODIFIED RUBEY'S LAW

In its original form, Rubey's law predicts fall times that are longer than the fall times (for fall through 100 cm of water) measured with the sediment analyzer (Figure 1). *Rubey's* [1933] original formula is shown below.

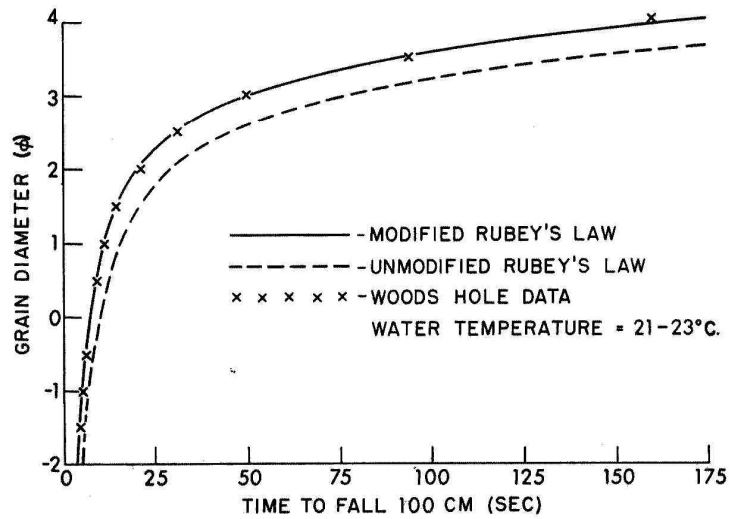

Fig. 1. Grain diameter versus time to fall 100 cm in water at a temperature of 21-23°C.

$$V = \frac{\sqrt{9\mu^2 + \frac{4}{3}R^3(\rho_{\text{particle}} - \rho_{\text{fluid}})g} - 3\mu}{R\rho_{\text{fluid}}} \tag{1}$$

Note that the curve for Rubey's law has exactly the same shape as that for the empiric data measured at Woods Hole [*Schlee*, 1966], but Rubey's law predicts fall times that are too long. The curve which nearly fits the data points is the curve generated by Rubey's law modified by empiric constants Z and W. This modified form of Rubey's law can be stated as follows:

$$V = \frac{\sqrt{9Z^2\mu^2 + \frac{4}{3}WR^3(\rho_{\text{particle}} - \rho_{\text{fluid}})g} - 3Z\mu}{WR\rho_{\text{fluid}}} \tag{2}$$

where $Z = .622$ and $W = .5305$. A plot of fall velocities measured at Woods Hole versus fall velocities calculated with the modified form of Rubey's law shows an excellent correlation (Figure 2 and Table 1). The fall velocity for groups of sediment grains can now be computed for any fluid and for any temperature, provided that the particle density and radius and the fluid density and viscosity are known.

The following equation, which is a solution of the modified Rubey's law for particle radius, can be used with fall velocity data obtained from the Woods Hole sediment analyzer to determine particle size:

$$R = \frac{\sqrt{W^2V^4\rho^2_{\text{fluid}} + 32Z(\rho_{\text{particl}} - \rho_{\text{fluid}})\mu Vg} + WV^2\rho_{\text{fluid}}}{\frac{8}{3}(\rho_{\text{particle}} - \rho_{\text{fluid}})g} \tag{3}$$

DERIVATION OF MODIFIED RUBEY'S LAW

The two equations above can be derived by writing an equation to balance the forces acting on a spheric particle falling through a fluid at terminal velocity. At terminal velocity the forces are in balance since the particle is not accelerating. Therefore the force balance equation can be written as follows:

particle weight in fluid = viscous drag force + pressure drag force

particle weight in fluid $= \frac{4}{3}\pi R^3(\rho_{\text{particle}} - \rho_{\text{fluid}})g$

viscous drag force (Stokes' law) $= 6\pi R\mu V$

pressure drag force $= \dfrac{\pi R^2 V^2 \rho_{\text{fluid}}}{2}$

The pressure drag force is the product of the projected area of the particle πR^2 and the pressure acting on it due to its motion relative to the fluid $\rho_f V^2/2$ [*Streeter*, 1966]. The similarity of shape between the curves for unmodified Rubey's law and the data (Figure 1) led me to believe that modification of the viscous drag and the pressure drag by empirically determined constants might lead to an accurate solution for fall velocity. These constants probably account for the lack of sphericity of the grains and nonideal behavior of the fluid. With this consideration the force balance equation is written as follows:

Particle weight in fluid = (Z) (viscous drag force) + (W) (pressure drag force)

or

$$\tfrac{4}{3}\pi R^3(\rho_{\text{particle}} - \rho_{\text{fluid}})g = Z(6\pi R\mu V) + W(\pi R^2 V^2 \rho_{\text{fluid}}) \qquad (4)$$

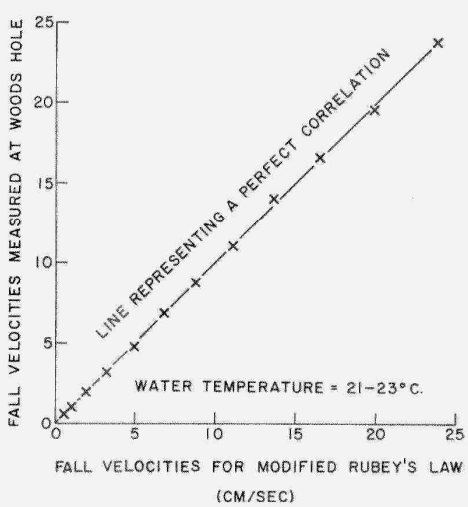

Fig. 2. Fall velocities of particles measured at Woods Hole by the Rapid Sediment Analyzer versus fall velocities calculated with the modified form of Rubey's Law. The straight line is the line (at 45°) which would represent a perfect correlation between the two sets of data.

TABLE 1. Fall Velocities of Particles Measured at Woods Hole by the Rapid Sediment Analyzer versus Fall Velocities Calculated with the Modified Form of Rubey's Law.

Woods Hole Velocity (cm/sec)	Modified Rubey's Law Velocity (cm/sec)	Grain Diameter ϕ
23.810	23.737	−1.500
19.608	19.826	−1.000
16.667	16.483	−.500
14.085	13.597	0.
11.111	11.072	.500
8.850	8.822	1.000
6.897	6.782	1.500
4.762	4.925	2.000
3.226	3.291	2.500
2.000	1.989	3.000
1.064	1.099	3.500
.625	.574	4.000

The factor of ½ in the pressure drag term is absorbed into the coefficient W to retain parallelism with the equation derived by *Rubey* [1933].

Equation 4 can now be simplified and solved for either particle radius or velocity by the quadratic formula yielding equations 3 and 2, respectively.

EVALUATION OF THE COEFFICIENTS Z AND W

As indicated in Figure 1 the fall velocity of particles of a diameter of -2ϕ to 0ϕ is controlled almost entirely by pressure drag. Particles of diameter 3ϕ to 4ϕ are controlled almost entirely by viscous drag (Stokes' law). In the intermediate range both drag forces are important. The CDC 6600 computer at the University of Texas at Austin was used to determine Z and W by trial and error in the following manner. Equation 4 above was solved separately for Z and for W in terms of all of the other variables. A value was assumed for W, then Z's were calculated for known diameters and their corresponding fall velocities in the 3ϕ to 4ϕ size range. The average of these Z values was then used to calculate a new value for W based on diameters and the corresponding velocities in the -2ϕ to 0ϕ range. This process of successive approximations was continued until both Z and W converged to constants. These values were then substituted into equation 2 and used to compute the modified Rubey's Law curve

(Figure 1). The close fit of the raw data from Woods Hole and the modified curve indicate that the values for constants Z and W are satisfactory.

CONCLUSIONS

Rubey's law modified with two empiric constants very accurately predicts fall velocities for groups of sand grains measured with the Woods Hole rapid sediment analyzer. This modified formula can now be used for calculation of particle size with fall time data from the sediment analyzer. The advantage of this method rather than the use of empiric data for particle size determination is that the determination can be made regardless of the water temperature in the settling tube, the viscosity of the fluid in the tube, or the relative densities of the fluid and the particles, as long as these values are known. This modified formula may also prove useful in predicting the fall velocity of groups of sediment grains in transport in such natural systems as river and wave transport.

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